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Computing the continuous-spectrum linearised bounded standing wave on a plane bed of arbitrary slope

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Abstract. The problem is revisited of continuous-spectrum small-amplitude bounded standing waves over a plane beach of arbitrary incline. Three documented approaches are compared and discussed *vis-à-vis* computational difficulties and this reveals, in particular, that the computation in established models, for particular beach angles, is optimised by seeking solutions to a functional integro-difference equation in a L_2 space. This simplification is found to be impossible for general beach angles and developed instead is an alternative method initially based on Cauchy Principal value convergence of the inversion integral but improved (by linear combination of solutions) and developed into a comprehensive computational package which is validated by three fundamental tests. A single f77 routine is provided for the user through a web-link. Comparisons with established but hitherto uncomputed methods are found numerically to favour the present development. Examples presented include both very steep and very shallow beaches and both potentials and velocities are calculated and displayed throughout the water column.

Key words: Bromwich contour, functional difference equation, Gamma function, Maliuzhinets' function, residue computation, W-transformation.

1. Introduction

Obliquely incident monochromatic waves of circular frequency ω on a plane incline of slope $\tan\alpha$ represents one of the classic eigenvalue problems in basic linearized water-wave theory. Ursell [1] has labelled as a 'mixed spectrum' the situation that prevails for this problem in so far as the eigenvalue problem for the wavenumber k has a discrete spectrum given by $\omega^2 = gk\sin(2n+1)\alpha; n \in \mathbf{N}$ if $k > \omega^2/g$ (Ursell's edge waves) and a continuous spectrum if $k < \omega^2/g$. Most readers will have encountered the latter (which are the subject of this paper), at some time or other, in, for instance, the description by Peters [2] also fully discussed in Stoker's book [3, pp. 95–109] with only slight modifications. However, there have been other descriptions for this problem, on arbitrary slope, in particular by Roseau in a series of papers, *e.g.* [4], and somewhat later by Lauwerier [5]. The present author in [6] revisited the problem to construct and compute solutions in terms of inverse Kontorovich-Lebedev transforms but was only able to do so for the most simple cases of beach angles of the type $\alpha = \pi/2M, M \in \mathbf{Z}$ (hereafter referred to as 'very simple' angles with the understanding that 'simple' angles are of the somewhat more general type $\alpha = p\pi/2M, (p, 2M) = 1$). The published material on other than very simple beaches appear to have remained purely theoretical to this date; the present author is certainly unaware of any published attempts at computations. Indeed, in recent unpublished work, Bruce [7] shows some of the considerable difficulties involved in computing, for example, the solution of Peters [2] even for very simple beaches. In [7] Bruce also computed the solutions for a number of special cases (and these included computation of the singular standing wave). However, even the case $M = 2$ is shown to present a real challenge using Peters's model, although it was shown by Roseau [4], almost at the same time, that the bounded wave could be expressed in closed form in terms of simple exponentials. It will be assumed that readers are reasonably familiar with these earlier works and some of their limitations.

Whilst the solutions on very simple beaches are readily computed and arguably infer the 'flavour' of solutions on non-simple beaches, the main interest in the computations is more often than not in a secondary exploitation of the solution, perhaps in a second-order model looking at steady currents, higher harmonics or set-down or undertow calculations (see *e.g.* [8]). In situations such as these, where large cancellations may occur, it is essential to be able to compute with significant accuracy, the first-order model and to be able to do so regardless of the angle of the beach or the angle of wave attack.

The main objective in this work, therefore, is to construct and to compute a solution for all beach angles which, above all, automatically reduces to the closed-form expressions, now well known, for the cases when the beach angles are very simple. Many authors have used the 2-d normal-incidence solutions for their related investigations but they generally appear to avoid the 3-d oblique-incidence versions and it is presumed that the lack of a readily obtainable numerical evaluation could be a major reason for this. For example Minzoni and Whitham [9] used the 2-d solution in discussing generation of edge waves whilst more recently Blondeaux and Vittori [10] also used it in an investigation of nonlinear resonant modes on a beach. Thus, the view will be taken here that what is required as the ultimate goal is a basic routine from which the solution is easily obtained by non-specialist mathematicians and as such a specific target is to develop a model which can be computed by a single f77 program linked to appropriate (public domain) QUADPACK library routines; this forms an integral part of the present paper. It is believed that this could be of special interest to modellers working also with beach protection schemes where typically inclines are greater than 45°, this being the steepest beach that enjoys the closed-form expressions well-known in the literature. With coastal protection schemes in mind, the beach of 60° inclination is used as one of the test examples near the end of this work.

The layout of the paper is as follows. The defining equations are written in Section 2 and this is followed by a section discussing some well-known solutions with 3 chosen such being compared in respect of their associated integral transforms and difference equations (DE). Methods of solving these are discussed in Section 4 for various beach angles and this is followed by a section describing the full solution to the beach water-wave problem using a new approach. A new element is found to be the possibility of solving the DE either in a L_2 space or outside it. The former approach leads more directly to the finite expansions of potential (or wave height) in cases when these are available, whilst the latter approach has to be adopted for non-simple beach angles. Computations are discussed in Sections 6 through 8 and a number of appendices provided hopefully make the reading of the paper less interrupted by detail. For the reader with little interest in mathematical detail of solutions, the possibility exists of passing over Sections 3–5 together with the appendices and concentrating instead on computing the solution as discussed fully in Sections 6–8 and the web-linked f77-file at *www/lgu.ac.uk/cismres/xtra/prog3.f.* Section 9 contains a concluding discussion with emphasis on the reliability of the method and the possibility of future work to discuss the robust computation of the singular wave which would be required in a progressing wave description (see *e.g.* discussion in [3, pp. 72–75]).

2. Equations

The potential function Φ is expressed in the form $\Phi = \Re\mathfrak{e}(\phi e^{i\omega t})$ where ω is the circular frequency of the monochromatic waves. The fundamental field equation for ϕ, assuming a long-shore dependence of the form $e^{i\kappa Z}$ will then be

$$\triangle\phi - \kappa^2\phi = 0 \tag{1}$$

with the boundary conditions normally adopted for this problem (see *e.g.* [3, p. 96])

$$\frac{\partial\phi}{\partial\theta}(R, -\alpha) = 0; \quad \frac{1}{R}\frac{\partial\phi}{\partial\theta}(R, 0) = \phi(R, 0), \tag{2}$$

where $z = Re^{i\theta}$ in the usual cylindrical polar coordinates (Z is used for the long-shore coordinate). In the above, time can be made non-dimensional by the transformation $t = t'/\omega$. There being no physical length scale in the infinite wedge, lengths are made non-dimensional w.r.t. the wave-length assumed at infinity, namely g/ω^2. In the rest of this work, it is assumed that all lengths and time have been scaled in this way. Closure of the system requires statements to be made about the far-field asymptotics and the behaviour at the origin. It is well-known that a solution bounded at the origin and having $O(1)$ wave behaviour at infinity will be perfectly reflected. Specifically, following Stoker [3, p. 96] the field at infinity is proportional to $e^{imx+y}e^{i(\kappa Z+\omega t)}$ so that $m^2 + \kappa^2 = 1$. Thus $\kappa > 1$ provides the discrete-spectrum solution representing trapped edge waves first written by Ursell [1] (and later by Roseau [11]) whilst $\kappa < 1$ provides the continuous-spectrum solutions that form the subject of this study whose wave fronts will therefore be directed by the wave number $\mathbf{k} = (m, \kappa)$. The case $\kappa = 1$ gives the so-called cut-off modes at the critical beach angles $(\pi/6, \pi/10, \pi/14, \dots)$ which can be thought of as limiting waves of 'glancing' incidence angle approaching zero (see [12]).

3. Established methods of solution

The solution constructed by Peters [2], albeit mathematically elegant, is extremely hard to compute, even for the very simple beach angles. This has been thoroughly demonstrated by Bruce in unpublished work [7]. The solution of Lauwerier [5] (see Appendix A) also presents numerical difficulties and (understandably in view of the computational facilities at the time) neither of these authors were able to compute their solutions numerically. Interestingly, though, Lauwerier pointed out that he had been unable to reconcile his solution analytically with that of Peters, although there were similar characteristics in the two.

The pivotal element in any of the solutions mentioned here is the inevitable difference equation that has to be satisfied by the kernel of whichever integral transform the author adopts. In the case of Peters [2] this was a Laplace transform (although in his treatment, Stoker [3, p. 97] opts to go straight to the contour-integral representation that Peters eventually arrives at). Lauwerier uses a Fourier integral (essentially a Sommerfeld integral representation). Both these authors obtain a first-order difference equation, whilst in [6] the present author obtains a second-order equation using the inverse Kontorovich-Lebedev transform (KLT). It is of interest to note, through the expression

$$2K_{is}(t) = \int_{-\infty}^{\infty} \exp(-t\cosh\theta - is\theta)d\theta$$

for the modified Bessel function, that the KLT can be regarded as a Laplace p-transform followed by a mapping $p = \cosh\theta$ and finally a Fourier transform in the θ-plane. The point

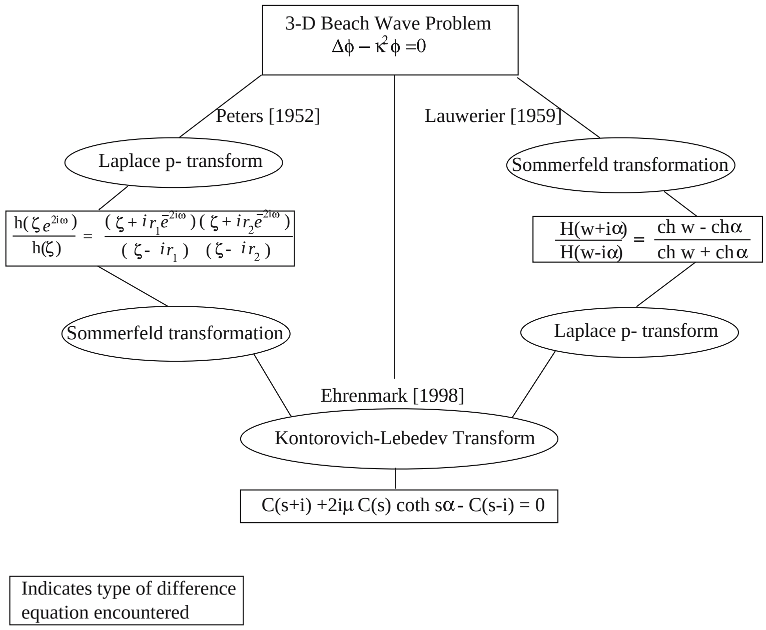

Figure 1. Schematic representation of 3 chosen solutions and their associated difference equations; in each case α is the beach angle.

about this observation is the illumination that Peters and Lauwerier evidently used different complementary KLT component transforms. If Peters used the 'front-end' of the KLT as defined here, then Lauwerier used the 'rear-end'.

Figure 1 shows schematically the relationships between the three types of solutions discussed and the types of equations occurring in the various approaches. This might offer some explanation why the two solutions were found hard to reconcile.

A functional first-order difference equation that is typical of that experienced in the above can be taken from the study by Williams [13] on electromagnetic diffraction in a wedge. Williams solves his equation (similar equations will be written in the next section) with the help of the double gamma function introduced by Barnes [14] and only a year before Williams's work, Maliuzhinets [15, 16] had defined a function, now known as Maliuzhinets's function $M_\beta(s)$, as a solution $f = M_\beta$ of

$$\frac{f(s+2\beta)}{f(s-2\beta)} = \cot\left\{\frac{1}{2}\left(s+\frac{1}{2}\pi\right)\right\} \tag{3}$$

which is analytic in $-2\beta - \pi/2 < \mathfrak{Re}(s) < 2\beta + \pi/2$ and satisfies $f(0) = 1$. Solutions to Williams's and other such related equations are then readily expressed in terms of Maliuzhinets functions and the advantage of such a representation is that a relatively simple integral expression is available for $M_\beta(s)$, namely

$$M_\beta(s) = \exp\left\{-\frac{1}{2}\int_0^\infty \frac{\cosh(xs)-1}{x\cosh(\frac{1}{2}x\pi)\sinh(2\beta x)}\,\mathrm{d}x\right\}. \tag{4}$$

Williams made the observation that Peters's solution was extremely difficult to compute because of the analytic continuation formulae required in various sectors. The test of time has substantiated this view and, with the exception of the recent unpublished work [7], there appear to have been no attempts made to perform evaluation from the original expressions given by Peters. Williams held the view that the infinite-product expansions would be easier to deal with and so these are written in the present work for the beach problem and used to develop the numerical strategy.

The major objective of this work, however, is to achieve a model which can be computed for arbitrary slope and incidence angles but which, nevertheless, reduces directly to the closed forms available for very simple slope angles. There is no doubt that the latter expressions are most easily obtained using the KLT approach (see [6]) but, on the other hand, the solution of the second-order difference equation for non-simple slopes does seem to lead back to the Lauwerier-type approach where the solution itself is expressed as a Sommerfeld integral. This approach is taken also by other authors, *e.g.* [17]. To reflect this, the intention is to proceed with the KLT formulation and allow the method of construction of a solution to the difference equation to lead us back to the Sommerfeld expression for the full solution. This helps explain other questions such as why Peters's solution, for example, is comparatively hard to express in simple terms for even the simplest of the very simple beach angles $\pi/4$. We note similarly, that Lauwerier's solution can be reduced to a finite sum of exponentials for very simple beaches, but only after a considerable amount of manipulation (the simplest case $\alpha = \pi/4$ is illustrated in Appendix A).

4. Solutions of difference equations

As an alternative to the method of using an integral expression such as the Maliuzhinets function, one can follow Williams [13] and use Barnes's double gamma function. In this case, the need to calculate an inner integral before the final outer integration for the solution can proceed, will be replaced by the need to evaluate an infinite product of gamma functions. An account of both methods is given here with appendices used prudently to minimise disruption to the text, but tests have shown that numerical computation of the infinite product referred to is fraught with difficulties. Thus, direct inversion of this method is not attempted.

By expressing a solution to system (1) and (2) in the form

$$\phi = \int_0^\infty A(s) K_{\mathrm{i}s}(\kappa R) \cosh s(\theta + \alpha)\ \mathrm{d}s, \tag{5}$$

the difference equation that has to be satisfied by the (even) function $A(s)$ is

$$A(\mathrm{i}p + \mathrm{i}) \sin(p+1)\alpha - 2\mu A(\mathrm{i}p) \cos p\alpha - A(\mathrm{i}p - \mathrm{i}) \sin(p-1)\alpha = 0, \tag{6}$$

where $\mu = 1/\kappa$ (see [6]). A solution to this is proposed in the form

$$A(p) = \lim_{X \to \infty} \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}X}^{c+\mathrm{i}X} e^{xp} f(x) \mathrm{d}x, \tag{7}$$

so that, with c suitably chosen, f is effectively a Laplace transform of A. Inserting (7) into (6) and making some rearrangement and the assumption (to be verified *a posteriori*) that the contour can be laterally dragged a distance 2α without interference from poles, we find that

$$\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} e^{\mathrm{i}p\xi} f(\xi - \alpha)\{e^{\mathrm{i}\xi} - 2\mathrm{i}\mu - e^{-\mathrm{i}\xi}\} \mathrm{d}\xi = \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} e^{\mathrm{i}p\xi} f(\xi + \alpha)\{e^{\mathrm{i}\xi} + 2\mathrm{i}\mu - e^{-\mathrm{i}\xi}\} \mathrm{d}\xi. \tag{8}$$

The procedure normally used by authors is to argue that this is satisfied by the difference equation

$$\frac{f(\xi + \alpha)}{f(\xi - \alpha)} = \frac{\sin \xi - \mu}{\sin \xi + \mu} \tag{9}$$

and Lauwerier obtained essentially this same equation [5, Equation 4.3], as might be expected from the scheme shown in Figure 1. Incidentally, whilst the focus in the present work is on the construction of the bounded standing wave, readers may note that the logarithmically singular standing wave (required for the construction of progressing waves) may be determined by seeking $A(s)$ in the alternative form $A^*(s)\coth\pi s$ where A^* is now of odd parity in s (see [6]).

4.1. A vertical cliff

The simple case $\alpha=\pi/2$ is often of interest to see how methods might work in more general cases. The general solutions to this problem appear to originate from Weinstein [18] but are also documented by Stoker [3, Section 5.3] and recovered by the author using the KLT in [6]. There it is found that the bounded solution

$$\phi=\mathrm{e}^{R\sin\theta}\cos(R\sqrt{(1-\kappa^2)}\cos\theta)$$

is given by the choice $A(s)=\cos\sigma s$ where $\cosh\sigma=1/\kappa=\mu$. The inverse of formula (7) gives the equivalent

$$f=\frac{x}{x^2+\sigma^2};\ \ \mathfrak{Re}x>\sigma,$$

which is not a solution of the functional Equation (9). Instead, Equation (8) is satisfied by the observation that each side is an integral of an entire function (simple poles being removable) and the Bromwich contours can be completed to give equality because each side is individually zero. The question then arises whether solutions to (9) will produce a solution to the problem. The form of $f(\xi)$ is unique for the bounded solution in the space of functions $f(c+\mathrm{i}x)\in L_2(-\infty,\infty)$ (see Titchmarsh [19, p. 67 *et seq.*]), so it is of some interest to compare the nature of $f(\xi)$ arising as a solution to (9). For the vertical cliff, this is easily demonstrated to be

$$f(\xi)=\Xi(\xi)\frac{\cos\frac{1}{2}(\xi+\mathrm{i}\sigma)}{\sin\frac{1}{2}(\xi-\mathrm{i}\sigma)},$$

where $\Xi(\xi+\alpha)=\Xi(\xi-\alpha)$ so that the solution integral for $A(s)$, Equation (7), converges only as Cauchy principal value at infinity when Ξ takes (as anticipated) a constant value.

4.2. Solution for general angles

In previous unpublished work [20], the author, unaware of the work of Maliuzhinets, introduced a function $B_k(s)$ as a solution to the difference equation

$$B(s+1)=B(s)\tan s\alpha. \tag{10}$$

This solution was subsequently used in a number of works, *e.g.* [21], to aid the description of the 2-D beach problem. The expression given in [20] is effectively

$$B_k(s)=\Gamma(s)\exp\left[\int_0^\infty\frac{\mathrm{d}t}{t}\left\{\frac{2\mathrm{e}^{t/2}\sinh(s-\frac{1}{2})t}{(\mathrm{e}^{kt}+1)(\mathrm{e}^t-1)}-\left(s-\frac{1}{2}\right)\mathrm{e}^{-t}\right\}\right],\ -k<\mathfrak{Re}s<k+1, \tag{11}$$

where now and hereafter $\alpha=\pi/2k$. It was, however, later noted (by Lawrie in remark appended to [22]) that a closed form may be obtained when k is integer:

$$B_k(s)=2^{k-1}\sqrt{2\pi}\,\csc\pi s\prod_{j=0}^{k-1}\cos(s+j)\alpha,\ \ 0<\mathfrak{Re}(s)<1. \tag{12}$$

Note also that B_k satisfies the same 'folding' formula as the gamma function, namely

$$B_k(s)B_k(1-s)=\frac{\pi}{\sin\pi s}. \tag{13}$$

It is clear that $B_k(s)$ as defined by Equation (11) is related to Maliuzhinets's functions and after a small amount of manipulation using Kummer's result [23, p. 250], we can express this relationship by

$$B_k(s)=\left\{\frac{\pi}{\sin\pi s}\frac{M_\beta(\frac{\pi}{2}+(s-\frac{1}{2})4\beta)}{M_\beta(\frac{\pi}{2}-(s-\frac{1}{2})4\beta)}\right\}^{\frac{1}{2}};\ \beta=\pi/4k,\ \ 0<\mathfrak{Re}(s)<1. \tag{14}$$

Given that Equation (9) can be rewritten

$$\frac{f(\xi+\alpha)}{f(\xi-\alpha)}=\frac{\tan\frac{1}{2}(\xi-\gamma)}{\tan\frac{1}{2}(\xi+\gamma)}, \tag{15}$$

where $\sin\gamma=\mu=\cosh\sigma$, it is straightforward to express a solution in terms of B_k functions. It will be convenient to modify the notation of f to indicate its dependance also on the parameter γ in which case that solution is written

$$f(\xi,\gamma)=\frac{B_k(\frac{\xi-\gamma}{2\alpha}+\frac{1}{2})}{B_k(\frac{\xi+\gamma}{2\alpha}+\frac{1}{2})}. \tag{16}$$

Note, with the help of the formula

$$M_\beta(\pi/2+s)M_\beta(\pi/2-s)=M_\beta^2(\pi/2)\cos(\pi s/4\beta)$$

(see [17, Equations B4 and B5]) that

$$f(\pi/2,\gamma)=1.$$

An alternative way of expressing this solution is obtainable through the use of Barnes's double gamma function. The result (see Appendix B for some details) for which $f(\pi/2,\gamma)=1$ is

$$f(\tau,\gamma)=\prod_{m=0}^{\infty}\frac{\Gamma\left\{\frac{\alpha-\gamma-\tau+(2m+2)\pi}{2\alpha}\right\}\Gamma\left\{\frac{\alpha+\gamma-\tau+(2m+1)\pi}{2\alpha}\right\}\Gamma\left\{\frac{\alpha+\gamma+\tau+(2m+1)\pi}{2\alpha}\right\}\Gamma\left\{\frac{\alpha-\gamma+\tau+(2m)\pi}{2\alpha}\right\}}{\Gamma\left\{\frac{\alpha-\gamma-\tau+(2m+1)\pi}{2\alpha}\right\}\Gamma\left\{\frac{\alpha+\gamma-\tau+(2m+2)\pi}{2\alpha}\right\}\Gamma\left\{\frac{\alpha-\gamma+\tau+(2m+1)\pi}{2\alpha}\right\}\Gamma\left\{\frac{\alpha+\gamma+\tau+(2m)\pi}{2\alpha}\right\}}. \tag{17}$$

It is evident that this form of solution is more useful in discussing the singularities of the function $f(\tau,\gamma)$. In fact, poles and zeros of the function $B_k(s)$ were discussed in [24] but it can readily be seen (since $\mathfrak{Re}\gamma=\pi/2$) from Equation (17) that there is a strip of regularity of $f(\tau,\gamma)$ given by $\pi/2-\alpha<\mathfrak{Re}(s)<3\pi/2+\alpha$. This is entirely consistent with the findings in [24] from which it can be confirmed that $B_k(s)$ has zeros at $s=-k, 2k+1$ and no others in between and that it has simple poles at $s=0, k+1$ and no others in between. The observation is important because the derivation of the formal solution required the existence of a strip of analyticity of width greater than 2α so that the two sides of Equation (8) can be brought together in the manner achieved using Cauchy's theorem. This still requires a discussion of the vanishing of the aggregate of two contributions on $\mathfrak{Im}(\xi)=\pm X$ in the limit $X\to\infty$. Finally, it is noted here that if $A(s)$ satisfies Equation (6), then so does $A(-s)$ and so we can replace Equation (7) by

$$A(p)=\lim_{X\to\infty}\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}X}^{c+\mathrm{i}X}f(x)\cosh px\,\mathrm{d}x. \tag{18}$$

5. Solution to the water-wave problem

5.1. Very simple beach angles

For the beach of 45° inclination ($k=2$), the author obtained [6]

$$A(s)=d_1\cosh s(\pi/4+\mathrm{i}\sigma)+c.c.$$

where the constant d_1 would be chosen to give the required wave amplitude at infinity. This form for A can be recovered by replacing the line of integration $\mathfrak{Re}(x)=c$ by a closed contour round poles at $\pi/4\pm\mathrm{i}\sigma$. A form of f which does this is readily seen to be

$$f=\frac{d_1}{x-x_1}+\frac{\overline{d}_1}{x-\overline{x}_1},$$

where $x_1=\gamma-\alpha$. This again is the L_2 form for f (as in the case of the cliff) but let us take a step back and examine how, for all very simple beaches, this form might be derived through the present *ansatz*, essentially Equation (2). The arguments leading to (9) effectively take us through a stage where we require

$$\oint\cosh xs\{f(x-\alpha)(\sin x-\mu)-f(x+\alpha)(\sin x+\mu)\}\,\mathrm{d}x=0 \tag{19}$$

for some closed contour which can be translated a distance 2α parallel to the real axis without traversing singularities. Instead of equating the brace to zero (which leads directly to (9)) we simply look for the integrand to be holomorphic inside the contour. Suppose f has simple poles at $x=x_j$, $j=1,2,\ldots,m$ and write

$$f(x)=\sum_{j=1}^{m}\frac{b_j}{x-x_j}+h(x),$$

where h is holomorphic. Application of (19) then yields

$$\oint\cosh xs\sum_{j=1}^{m}\left\{\frac{b_j(\sin x+\mu)}{x-(x_j-\alpha)}-\frac{b_j(\sin x-\mu)}{x-(x_j+\alpha)}\right\}\mathrm{d}x=0.$$

The points x_j are now chosen so that $x_j+\alpha=x_{j+1}-\alpha$. In this way we can choose also

$$b_j(\sin(x_j+\alpha)-\mu)=b_{j+1}(\sin(x_j+\alpha)+\mu),\quad j=1,2,\ldots,m-1.$$

Then, Equation (19) is guaranteed, since the first term of the first sum and the last term of the last sum are regular because of the zeros of the numerator. This then fully generates the L_2 solutions for all very simple beach angles. Explicitly the points x_j are

$$x_1=-\frac{\pi}{2}+\alpha,\ldots,x_{j+1}=x_j+2\alpha,\ldots,x_m=\frac{\pi}{2}-\alpha$$

and the corresponding solution for $A(s)$ is as given by the author in [6, Equation (5.2)].

5.2. The case $k = 2$

It will be worthwhile to discuss the beach of unit gradient a little further. For integer values of k there will always be cancellation of the gamma functions in (17) and for the case $k = 2$, using the abbreviations $z_{\pm} = (\tau \pm \gamma)/2\alpha$, we have

$$f(\tau,\gamma) = \frac{(3+2z_-)(1+2z_-)}{(3+2z_+)(1+2z_+)} \prod_{m=0}^{\infty} \frac{\left\{(4m)^2 - (\frac{1}{2}+z_+)^2\right\}\left\{(4m)^2 - \left(\frac{3}{2}+z_+\right)^2\right\}}{\left\{(4m)^2 - (\frac{1}{2}+z_-)^2\right\}\left\{(4m)^2 - \left(\frac{3}{2}+z_-\right)^2\right\}},$$

which we may readily simplify, using the infinite product

$$\sin\left(\frac{1}{2}\pi z\right) = \frac{\pi z}{2} \prod_{r=1}^{\infty} \left\{1 - \frac{z^2}{4r^2}\right\},$$

to

$$f(\tau,\gamma) = \frac{\sin\left(\frac{\pi}{8} + \frac{1}{2}(\tau+\gamma)\right)\sin\left(\frac{3\pi}{8} + \frac{1}{2}(\tau+\gamma)\right)}{\sin\left(\frac{\pi}{8} + \frac{1}{2}(\tau-\gamma)\right)\sin\left(\frac{3\pi}{8} + \frac{1}{2}(\tau-\gamma)\right)}. \tag{20}$$

This solution can also be recovered using the representation (16) most easily combined with the convolution result $B_k(s)B_k(1-s) = \pi/\sin(\pi s)$.

In order best to express the full solution, we note the result

$$\int_0^{\infty} 2\cosh s\left(\frac{\pi}{2} - y\right)\cosh s(\theta+\alpha) K_{is}(\kappa R)\ \mathrm{d}s = \pi \mathrm{e}^{-\kappa R \sin y \cos(\theta+\alpha)} \cosh\{\kappa R \cos y \sin(\theta+\alpha)\};$$

see [25, p. 245]. Combine this with the closed contour integral for $A(s)$ and the *ansatz* integral for the solution, Equation (5), change orders of integration and there follows the well-known result for the beach of unit gradient,

$$\phi = -\mathrm{i}\pi\mu^2\{\mathrm{e}^{Y}(T\cos TX - \sin TX) + \mathrm{e}^{-X}(T\cos TY + \sin TY)\}$$

in terms of Cartesian coordinates $(X, Y) = R(\cos\theta, \sin\theta)$ where $T = \tanh\sigma$. Similar expressions for all very simple beach angles are given by the author as [6, Equation (5.3)]. It is not the intention to reconsider these fully here; instead we now move on to describe a method of obtaining the solution for general (non-simple) beach angles.

5.3. The case of non-simple beach angles

Having derived a general solution of the functional equation in a form for which the solution integral will only converge as a Cauchy principal value at infinity, it is convenient to make some observations in order to recover an absolutely convergent integral. These observations are readily seen from Equation (17) but are also established in Appendix C by use of the integral expression (11) and can be listed thus:

$$(i)\quad \frac{f(\tau,\overline{\gamma})}{\cos\frac{\pi}{2\alpha}(\tau-\gamma)} = \frac{f(\tau,\gamma)}{\cos\frac{\pi}{2\alpha}(\tau-\overline{\gamma})}$$

$$(ii)\quad \frac{f(-\tau,\gamma)}{\cos\frac{\pi}{2\alpha}(\tau-\gamma)} = \frac{f(\tau,\gamma)}{\cos\frac{\pi}{2\alpha}(\tau+\gamma)}$$

$$(iii)\quad \frac{f(-\tau,\overline{\gamma})}{\cos\frac{\pi}{2\alpha}(\tau-\gamma)} = \frac{f(\tau,\gamma)}{\cos\frac{\pi}{2\alpha}(\tau+\overline{\gamma})}$$

so that, in particular

$$\frac{f(\tau,\gamma)-\nu f(\tau,\overline{\gamma})}{f(-\tau,\gamma)-\nu f(-\tau,\overline{\gamma})} \sim -1,\, \tau = \mathrm{i}y,\, y \to \pm\infty$$

provided that

$$\nu = \frac{\cos\frac{\pi\gamma}{2\alpha}}{\cos\frac{\pi\overline{\gamma}}{2\alpha}}.$$

Since both $f(\tau,\gamma)$ and $f(\tau,\overline{\gamma})$ provide solutions, inverting the KLT as above allows construction of the further solution

$$\phi = \varsigma_\infty \int_C \{\mathrm{e}^{-\kappa R\cos(\tau+(\theta+\alpha))} + \mathrm{e}^{-\kappa R\cos(\tau-(\theta+\alpha))}\}\Upsilon(\tau,\gamma)\ \mathrm{d}\tau \tag{21}$$

where

$$\Upsilon(\tau,\gamma) = \cos\frac{\pi\overline{\gamma}}{2\alpha} f(\tau,\gamma) - \cos\frac{\pi\gamma}{2\alpha} f(\tau,\overline{\gamma}),$$

ς_∞ is a constant chosen to give unit amplitude at infinity and C is a contour in the r.h. half plane asymptotic to the imaginary axis at both ends, cutting the real axis in the interval $\pi/2 < \Re\mathfrak{e}(\tau) < 3\pi/2$ and passing to the right of the points $\pi/2 \pm \mathrm{i}\sigma$ (see Figure 2). Note that this is now in the more usual Sommerfeld integral form but that the construction in terms of $\Upsilon(x,\gamma)$ extends the absolute convergence of the integral uniformly to include the point $R=0$. This is a deficiency in Lauwerier's solution.

At this stage, rigorous verification of the solution is straightforward. Differentiation under the integral sign is justified by dominated convergence and the only move requiring some

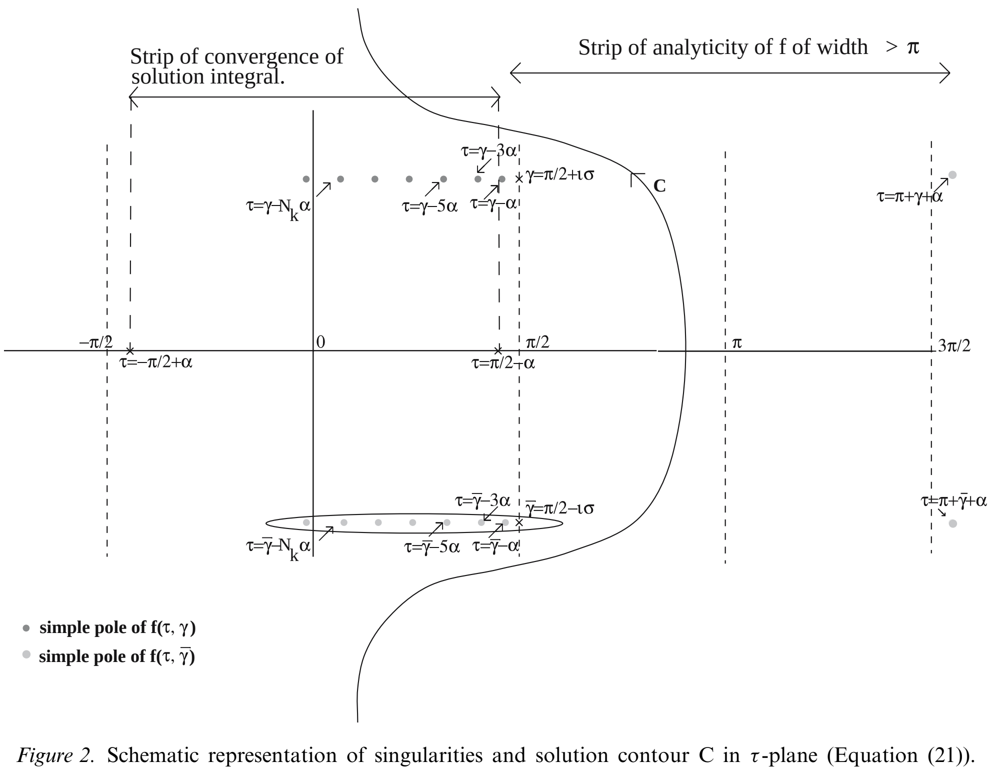

Figure 2. Schematic representation of singularities and solution contour C in τ-plane (Equation (21)).

further justification is in shifting the contours to satisfy the surface boundary condition. Setting $\tau = c + \mathrm{i}y$, we are faced with one contour integral having the convergence factor $\mathrm{e}^{-\kappa R \cosh y \cos(c-\alpha)}$ and needing to be 'dragged' a distance 2α parallel to the real axis to match its partner term. Thus, the requirement is simply tantamount to $\alpha \leq \pi/2$ where equality can be tolerated by principal-value arguments as $|\tau| \to \infty$. The present theory would therefore appear to be invalid for the case of an overhanging cliff.

6. Computing the solution

It will be demonstrated below that the solution can be represented by a residue sum together with an infinite integral which vanishes identically for simple beaches. It will also be demonstrated (see 6.3, particularly 6.3.2, below) that if non-simple beaches are very shallow, this residue sum can become very large and accurate final evaluation relies on cancellation of this with almost equal and opposite large values from the numerical integration. In these cases, that mode of computation is impractical and will be replaced by a more stable numerical integration on an alternative contour which does not embrace all the poles. It turns out that embracing just the two poles with greatest real part is useful because these two provide the field at infinity.

Let

$$g(\tau|\underline{R}) = \mathrm{e}^{-\kappa R \cos(\tau+(\theta+\alpha))} + \mathrm{e}^{-\kappa R \cos(\tau-(\theta+\alpha))};$$

then it is easy to show that

$$\begin{aligned}&\operatorname{cosech}(\pi\sigma/\alpha)\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} g(\tau|\underline{R})\Upsilon(\tau,\gamma)\ \mathrm{d}\tau\\ &= 2\sin\left(\frac{\pi^2}{2\alpha}\right)\left|\cos\frac{\pi\gamma}{2\alpha}\right|^2\int_0^{\mathrm{i}\infty} g(\tau|\underline{R})f(\tau,\gamma)\frac{\cos(\frac{\pi\tau}{2\alpha})}{\cos\frac{\pi}{2\alpha}(\tau+\gamma)\cos\frac{\pi}{2\alpha}(\tau-\overline{\gamma})\cos\frac{\pi}{2\alpha}(\tau+\overline{\gamma})}\ \mathrm{d}\tau\end{aligned}$$

where $k = \pi/2\alpha$, $\gamma = \pi/2 + \mathrm{i}\sigma$, and $\cosh\sigma = 1/\kappa$ (the waves at infinity making an angle $\sin^{-1}\kappa$ with the shore-line normal). Thus, using Equation (16), we can express a solution having unit amplitude at infinity in the alternative form

$$\phi = \Phi_{\mathrm{res}} + 2\mathrm{i}\varsigma_\infty \sinh 2k\sigma \sin k\pi\,|\cos k\gamma|^2 \int_0^{\mathrm{i}\infty} \frac{g(\tau|\underline{R})f(\tau,\gamma)\cos k\tau}{\cos k(\tau+\gamma)\cos k(\tau-\overline{\gamma})\cos k(\tau+\overline{\gamma})}\ \mathrm{d}\tau \qquad (22)$$

where Φ_{res} denotes the appropriate sum of residues at the poles of Υ with non-negative real part in $\Re\mathfrak{e}(\tau) < \pi/2$.

The significance of Equation (22) is the multiplier $\sin k\pi$ which ensures that the 'remainder' term will vanish identically for the very simple beach angles when k is integer. We therefore have a general solution which automatically reduces to the finite sum already well-known (*e.g.* [11]) for these angles.

In the non-simple cases the integral on $[0, \mathrm{i}\infty]$ (which converges exponentially for all values of R) is added and in the case where poles fall on the imaginary axis, this integral is interpreted as a principal-value integral and only half the corresponding residues are taken.

6.1. Computing the residues

A glance at the numerators in Equation (17) shows that there are four groups of singularities (which are all simple poles if α is an irrational multiple of π), two of the groups have no poles to the right of the line $\mathfrak{Re}(\tau)=\pi/2-\alpha$, whilst the other two have none to the left of the line $\mathfrak{Re}(\tau)=3\pi/2+\alpha$. Specifically, with $(m,N)=0,1,2,\dots,$

$$\begin{aligned}
&\Gamma_1: \ \tau=\alpha+\gamma+(2m+1)\pi+2N\alpha,\\
&\Gamma_2: \ \tau=\alpha-\gamma+(2m+2)\pi+2N\alpha,\\
&\Gamma_3: \ \tau=-\alpha-\gamma-(2m+1)\pi-2N\alpha,\\
&\Gamma_4: \ \tau=-\alpha+\gamma-(2m)\pi-2N\alpha,
\end{aligned}$$

defines the four sets $\Gamma_j,\ j=1,2,3,4$ where $f(\tau,\gamma)$ has poles and, regardless of the arithmetic nature of α, those in the region $0<\mathfrak{Re}(\tau)<\pi/2$ emanate from the 'primary' set when $m=0$ and must therefore all be simple poles. This is an important observation since we want to pass the solution integral contour only across that region. The poles of Υ in the region arise from the primary set of Γ_4 and are at

$$\tau=\pi/2-\alpha\pm\mathrm{i}\sigma,\quad \pi/2-3\alpha\pm\mathrm{i}\sigma,\quad \pi/2-5\alpha\pm\mathrm{i}\sigma,\dots,\quad \pi/2-N_k\alpha\pm\mathrm{i}\sigma.$$

where N_k is the odd value of either $[k]$ or $[k-1]$.

By using the continuation formula $B_k(s+1)=B_k(s)\tan s\alpha$, we can write down the value of the contribution due to the sum of all the relevant residues thus:

$$\begin{aligned}
\Phi_{\mathrm{res}}=2\pi\mathrm{i}\Omega\varsigma_\infty\sum_{N=0}^{N_{\max}}\lambda_N\{\mathrm{e}^{-\kappa R\cos(\gamma+\theta-2N\alpha)}+\mathrm{e}^{-\kappa R\cos(\gamma-\theta-2(N+1)\alpha)}\}\\
\times\prod_{j=0}^{N-1}\frac{\cot(j-N)\alpha}{\cot(\frac{\gamma}{\alpha}+j-N)\alpha}+\text{c.c.}
\end{aligned}\tag{23}$$

where

$$\Omega=\frac{2B_k(1)}{B_k(\frac{\gamma}{\alpha})}\cos\left(\frac{\pi\overline{\gamma}}{2\alpha}\right),\quad \lambda_N=1,\ N<N_{\max};\ \lambda_{N_{\max}}=\frac{1}{2}$$

and $N_{\max}=\left[\frac{k-1}{2}\right]$ and it is further understood both that the product of cotangents takes the value unity when $N=0$ and that c.c. means that γ is replaced by $\overline{\gamma}$ in the previous term. Note that, writing $\Omega=\Omega_0\mathrm{e}^{\mathrm{i}\delta}$ the wave at infinity is given by

$$\Phi_{\mathrm{res}}^{(\infty)}=-4\pi\Omega_0\varsigma_\infty\sin(TR+\delta)$$

where $T=\tanh\sigma$, so inferring the choice $\varsigma_\infty=\frac{1}{4\pi\Omega_0}$ to give a unit amplitude wave at infinity. Note also that the introduction of λ_N validates the formula also for the case where there are poles on the imaginary axis (these are the Ursell edge wave critical slope angles $\pi/6,\pi/10,\dots$ (see [1])). In two examples below (see Section 6.3) are computed the sum of residues for the two cases (i) $\alpha=1^c$ and (ii) $\alpha=0.1^c$ with the respective corresponding values (i) $N_{\max}=0$ and (ii) $N_{\max}=7$. The computation of Ω now needs to be done using the integral expression (11) since k is irrational. However, the value of $B_k(1)$ is obtained easily by combining the 'folding' formula (13) with the difference relation (10) and taking the limit $s\to 0$. This yields $B_k(1)=\sqrt{\alpha}$. In order to develop a unit amplitude at infinity, there is a need to compute $|B_k(1)/B_k(\gamma/\alpha)|$. By using the results of [24, Appendix 1], it can readily be shown that

$$\frac{B_k(1)}{B_k(\rho)}=\exp\int_0^\infty\frac{\sinh(k-\rho)t\sinh(\rho-1)t}{t\cosh kt\sinh t}\,\mathrm{d}t;\quad 0<\mathfrak{Re}(\rho)<k+1.\tag{24}$$

With the help of [26, Art. 4.116(2)] it is possible to establish the closed-form result

$$|B_k(1)/B_k(\gamma/\alpha)| = \cosh k\sigma \left(\frac{\tanh k\sigma}{k\tanh\sigma}\right)^{1/2},$$

details of this are given in Appendix C. This enables the full solution to be rewritten

$$\phi = \Phi_{\text{res}} - \mathrm{i}\lambda \int_0^{\mathrm{i}\infty} \frac{g(x)f(x,\gamma)\cos kx}{\cos k(x+\gamma)\cos k(x-\overline{\gamma})\cos k(x+\overline{\gamma})}\ \mathrm{d}x \tag{25}$$

where

$$\lambda = \frac{\sin(k\pi)\sinh k\sigma|\cos k\gamma|}{2\pi}\left(\frac{k\tanh 2\sigma}{\tanh 2k\sigma}\right)^{\frac{1}{2}}.$$

6.2. Near-shore behaviour

An expression for the argument δ of Ω is given by

$$\delta = \arg\Omega = \arg\cos k\overline{\gamma} - \frac{1}{2}\int_0^{\infty} \frac{\sin\left(\frac{2\sigma t}{\alpha}\right)}{t}\left(\frac{\tanh kt}{\tanh t} - 1\right)\ \mathrm{d}t. \tag{26}$$

In the case of the very simple beach $k=2$, the integral remainder term vanishes identically (as it will for all very simple beaches) and the expression for Φ_{res} reduces to just one term of the sum (since $N_{\max}=0$) and is all that is required to produce the full solution. Nevertheless, it is not entirely straightforward to recover the classical result of shoreline wave amplification by a factor of $\sqrt{2}$ in the limit of normally incident waves ($\sigma\to\infty$). The difficulty stems from the need accurately to calculate $\operatorname{Arg}B_2(\gamma/\alpha))$ as $\sigma\to\infty$. This is tantamount to examining $\operatorname{Arg}B_2(\mathrm{i}\sigma)$ in the same limit and the expression (12) cannot be used since it is restricted to $\mathfrak{Re}(s)>0$. It is shown in [24], however, that, on $s=c+\mathrm{i}\tau$

$$B(s) = \sqrt{2\pi}\,\exp\left\{-\frac{1}{2}\pi|\tau| + \frac{\mathrm{i}\pi}{4}(2c-(k+1))\,\mathrm{sgn}\ \tau\right\}\left\{1+\mathrm{O}(\mathrm{e}^{-\pi|\tau|/k})\right\};\quad 0<c<1$$

It follows that $\operatorname{Arg}\,\Omega \to \frac{\pi}{4}(k+1)\mathrm{sgn}\ \tau$. This means that, after inserting the time factor and taking real parts, the two terms which contribute the asymptotic form as $R\to\infty$ amount to $\Phi_{\text{res}}^{\infty} = -4\pi\Omega_0\sin\omega t\cos(\kappa R\sinh\sigma+\delta)$ where $\Omega=\Omega_0\mathrm{e}^{\mathrm{i}\delta}$. Meanwhile, the value of Φ_{res} at the origin is given by $\Phi_{\text{res}}^{0} = 8\pi\Omega_0\cos\delta\cos\omega t$ thus giving, as required, the amplification $2|\cos\delta|$ in the limit of normal incidence.

In the next subsections are delivered results for some extreme cases *i.e.*, both shallow and steep beaches and both near normal and glancing incidence angles. In doing this, we will be show that the 'remainder' integral is generally not small compared to the residue contributions and near-shore it is often of the same order of magnitude.

6.3. Residue contributions

6.3.1. *Example 1*

For a wave on the beach of angle one radian, incident at an angle for which $\sigma=1$, *i.e.*, approx. 40°, it is found that $\Omega = 4{\cdot}10756 - 1{\cdot}56053\mathrm{i}$. The expression for wave height contributed by the residues is

$$\eta_{\text{res}} = \Omega\left\{\mathrm{e}^{\mathrm{i}\kappa R\sinh\sigma} - \frac{\cos(2\alpha-\mathrm{i}\sigma)}{\cosh\sigma}\mathrm{e}^{-\kappa R\sin(2\alpha-\mathrm{i}\sigma)}\right\} - \text{c.c.}$$

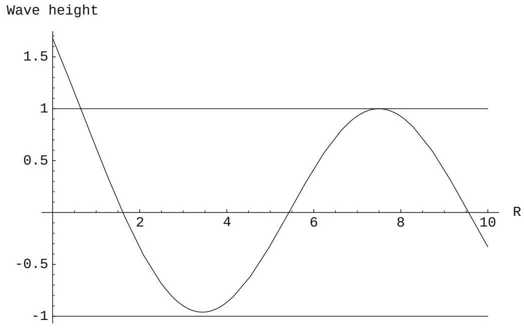

Figure 3. Contribution to wave height from residue sum; Case $\alpha=1, \sigma=1$ (angle of incidence $\approx 40°$).

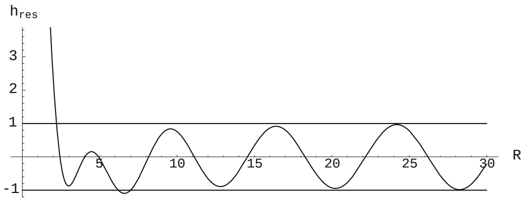

Figure 4. Contribution to wave height from residue sum; Case $\alpha = 0{\cdot}1, \sigma = 1$ (angle of incidence $\approx 40°$). Note the inordinately large values near the shore line.

which, for the chosen parameters, reduces to

$$\eta_{\text{res}} = \Omega\left\{e^{iTR} - (\cos 2 + iT \sin 2)e^{-R\sin 2}e^{iTR\cos 2}\right\} - \text{c.c.} \tag{27}$$

A graph of $i\eta_{\text{res}}$ is shown in Figure 3. This compares qualitatively as well as can be expected with the dashed contour in [6, Figure 2a] which is for 41 degree incidence on a 45° beach.

6.3.2. *Example 2*

For a shallow beach we take $\alpha=0{\cdot}1$ and retain $\sigma=1$; there follows similarly $\Omega=-6{\cdot}25382+10{\cdot}816i$ and an expression for wave height contributed by the residues is

$$\begin{aligned}\eta_{\text{res}} = \Omega \sum_{N=0}^{7} &\left\{ \sin(\gamma - 2N\alpha)e^{-\kappa R\cos(\gamma-2N\alpha)} - \sin(\gamma - 2(N+1)\alpha)e^{-\kappa R\cos(\gamma-2(N+1)\alpha)}\right\}\\ &\times\Pi - \text{c.c.}\end{aligned} \tag{28}$$

where

$$\Pi = \prod_{j=0}^{N-1} \frac{\cot(j-N)\alpha}{\cot(\frac{\gamma}{\alpha}+j-N)\alpha}$$

A similar graph of $i\eta_{\text{res}}$ for this case is shown in Figure 4. This illustrates more clearly the difficulties with using the residue method for very shallow beaches. When these shallow beaches are of the 'very simple' type, the cotangents (in the product expression) have a symmetry which suppresses their overall product. When the beach slope is not of this type, this cancellation does not happen with a consequence that very large values are encountered near the origin. It is well-known, for example, that the shore-line magnification is no greater than its normal incidence value $\sqrt{k}$ which for the beach of angle 0·1 is just 3·96 and this can be compared with the magnification of η_{res} which is 24·91. Evidently therefore the residue contributions are (in general) in no sense an approximation to the solution and the expression referred to earlier as 'remainder integral' will consequently be relatively large for small values of R. The remarks above lead to the conclusion that an alternative procedure is required to give robustness to the process and avoid this numerical instability for very gentle beach slopes.

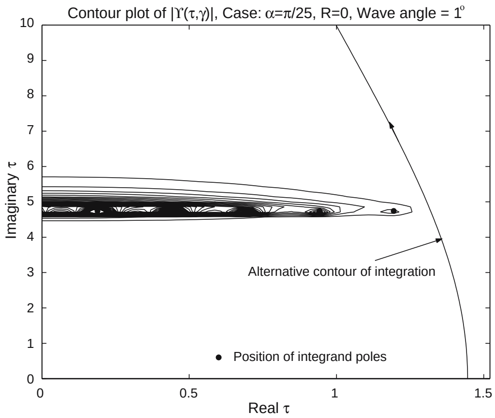

Figure 5. Modulus of integrand in Equation (30).

7. Computation of solution using an alternative contour

In order to avoid the difficulty of large cotangent products which occur for very shallow beaches, it will be necessary more directly to compute the solution on the given inversion contour. It is found advantageous, however, to include the contribution Φ_0 of the first two residues at the poles $\tau = \pi/2 - \alpha \pm i\sigma$, as these not only determine the wave field at infinity but also because the integrand will then be substantially better behaved on a contour passing to the left of these two poles. Denote such a contour C* by $\tau = X + iY$ where

$$X = (\pi/2 - \alpha)\mathrm{sech}\left(\frac{Y}{\sigma}\cosh^{-1}\frac{k-1}{k-2}\right), \quad Y \in [-\infty, \infty].$$

The contour C* is then contained in the strip $0 < \mathfrak{Re}(\tau) < \pi/2 - \alpha$ and is symmetric w.r.t. the real axis. It is also asymptotic to the imaginary axis and bisects the line joining the two poles at $\tau = \pi/2 - \alpha, \tau = \pi/2 - 3\alpha$. The full solution can then be written

$$\phi = \Phi_0 + \varsigma_\infty \int_{C*} \{e^{-\kappa R\cos(\tau + (\theta+\alpha))} + e^{-\kappa R\cos(\tau - (\theta+\alpha))}\}\Upsilon(\tau, \gamma)\ d\tau, \tag{29}$$

where, with δ defined by Equation (26), Φ_0 is given by

$$\Phi_0 = \mathfrak{Re}\{ie^{i\delta}(e^{-\kappa R\cos(\gamma+\theta)} + e^{-\kappa R\cos(\gamma-\theta-2\alpha)})\}$$

By combining the integrals along the respective parts of the path in the upper and lower half planes, we may readily simplify to the expression

$$\phi = \Phi_0 + \varsigma_\infty\left(\int_0^\infty (X' + i)g(X + iY|\underline{R})\Upsilon(X + iY, \gamma)\ dY + \text{c.c.}\right)$$

where X' denotes $\frac{dX}{dY}$. With the result in Appendix C_3, this can be written in the alternative form

$$\phi = \Phi_0 + \lambda^* \int_0^\infty \mathfrak{Re}\Bigg((X' + i)g(\tau|\underline{R})\left\{\frac{1}{\cos k\gamma\cos k(\tau-\gamma)} - \frac{1}{\cos k\overline{\gamma}\cos k(\tau - \overline{\gamma})}\right\} \times G(\tau, \sigma)\Bigg)\ dY, \tag{30}$$

where

$$\tau = X + \mathrm{i}Y, \quad (\lambda^*)^2 = \frac{k(s^2\tanh^2 k\sigma + c^2)\tanh 2\sigma}{16\pi^2\tanh 2k\sigma}; \quad (c,s) = (\cos,\sin)k\pi/2$$

and

$$G(\tau,\sigma) = \exp\int_0^\infty \frac{1}{t\sinh\alpha t}\left\{1 - \frac{\cosh t\tau}{\cosh\pi t/2}\cos\sigma t\right\}\,\mathrm{d}t.$$

For illustration, the absolute value of the integrand in the outer integral is displayed (over the semi-infinite strip $\mathfrak{Im}\tau > 0$ through contours for the case $\alpha = \pi/25$ when $R = 0$ and the incidence angle is just one degree. This shows, in particular, the difficulty with the 'residue + remainder integral' method as beaches become more shallow. The intensity of the poles increases as $\mathfrak{Re}\tau$ decreases and any path of integration contained near the imaginary axis is inevitably going to have to climb to and descend from a large height.

An expression can also be written for the velocity field $\underline{v}$. Recall that g is defined by $g(\tau|\underline{R}) = \mathrm{e}^{-\kappa R\cos(\tau+(\theta+\alpha))} + \mathrm{e}^{-\kappa R\cos(\tau-(\theta+\alpha))}$ so that an expression is readily obtained, *viz.*

$$\underline{v} = \nabla\Phi_0 + \varsigma_\infty\left(\int_0^\infty (X' + \mathrm{i})\nabla g(X + \mathrm{i}Y|\underline{R})\Upsilon(X + \mathrm{i}Y,\gamma)\,\mathrm{d}Y + \mathrm{c.c.}\right). \tag{31}$$

It should be noted that, despite the extra exponentially increasing (with Y) factor arising from the grad operator in the integrand, the overall convergence is largely unaffected, given that $|\nabla g|$ will be of order $\mathrm{e}^{Y-\iota\kappa R\cosh Y}$ as $Y \to \infty$, where $\iota = \min(\sin\alpha,\cos\alpha)$.

8. Results

8.1. Numerical procedure

The inner integral is first split into $\int_0^a + \int_a^\infty$, where a is suitably chosen so that the first integral is non-oscillatory and is therefore easily computed, with the help of an appropriate Taylor expansion, using QUADPACK routine *dqag*. The choice is governed by $a = \min(\pi/2\sigma, \pi/2Y)$. In the second integral, the term $1/t\sinh\alpha t$ is integrated separately by the routine *dqagi* (for semi-infinite interval) but the remaining part is now oscillatory and increasingly so as $Y \to \infty$. To deal with this difficulty, we will use the W-transformation (see [27]) which requires the computation of a sequence of finite integrals between the zeros of the controlling oscillatory factor. However, depending on whether $Y > \sigma$ or $Y < \sigma$, this controlling factor is either $\cosh\tau t$ or $\cos\sigma t$. This needs to be negotiated as does the related observation that, when $Y \approx \sigma$, part of this integrand is non-oscillatory and the W-algorithm would fail. Thus, we need to exclude, from this treatment, a small interval of the outer integration range, where instead the value of the inner integral is computed using *dqag*.

With regard to the outer integral, the routine *dqagi* (intended for infinite integrals) does not behave well for incidence angles of 30 degrees or beyond when beach slopes are comparatively small. Attempts to cure this by strategic adjustment of parameters *epsrel* and *epsabs* have not succeeded and it was therefore decided to let a truncation value for the routine be determined by the asymptotics of the integrand. Beyond this truncation value, an asymptotic term is used. Conventional arguments involving the residue theorem readily show that, provided $|\mathfrak{Re}(\tau)| < \alpha$ (so that equation (C2) may be used), then asymtotically as $\mathfrak{Im}(\tau) \to \infty$ we have

$$G(\tau,\sigma) \sim \cos k\tau\exp\left(\frac{2\cosh\sigma\mathrm{e}^{\mathrm{i}X-Y}}{\sin\alpha}\right).$$

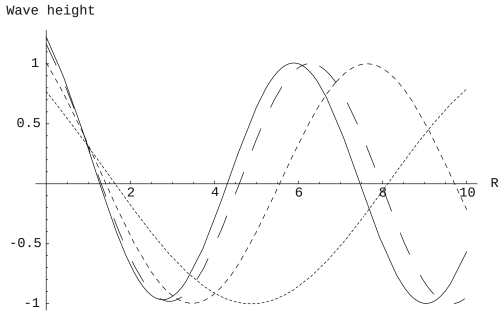

Figure 6. Wave heights for varying incidence; beach angle $\alpha=\pi/3$; angle of incidence, full line: 1°, broken line 21°, dashed line 41°, dotted line 61°.

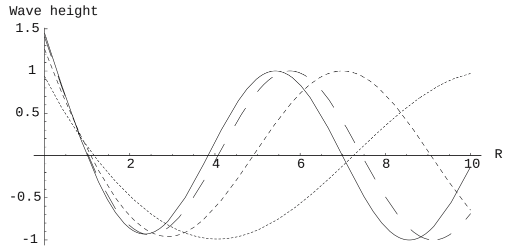

Figure 7. Wave heights for varying incidence; beach angle $\alpha=\pi/4{\cdot}2$; angle of incidence, full line: 1°, broken line 21°, dashed line 41°, dotted line 61°.

By examining also the next term (provided $\alpha<\pi/3$ this arises from poles at $t=\pm 3\mathrm{i}$), it is established that the relative error of neglecting this term is $<10^{-N}$ provided that $Y>Y_c=\max(Y_c^{(1)},Y_c^{(2)})$. Here $Y_c^{(1)}=\frac{1}{2}\log\left(\frac{10^N\cosh(3\sigma)\sin(\alpha)}{\cosh(\sigma)\sin(3\alpha)}\right)$ and $Y_c^{(2)}$ is chosen so that $X<\alpha$, *i.e.*, $Y_c^{(2)}=\sigma\frac{\cosh^{-1}k-1}{\cosh^{-1}\frac{k-1}{k-2}}$.

In order to establish the upper truncation value, the dominant asymptotics of the outer integrand is written. Because X may be zero (for integration on imaginary axis undertaken when $k<2{\cdot}3$) it is necessary to keep some extra terms. It is found that

$$\left\{\frac{1}{\cos k\gamma\cos k(\tau-\gamma)}-\frac{1}{\cos k\overline{\gamma}\cos k(\tau-\overline{\gamma})}\right\}G(\tau,\sigma)\sim\frac{\sinh 2k\sigma}{|\cos k\gamma|^2}\left(1+\frac{2\cosh\sigma\exp^{\mathrm{i}X-Y}}{\sin\alpha}\right)+\varepsilon,$$

where

$$\varepsilon=\mathrm{e}^{-2k(Y-\mathrm{i}X)}\left(\frac{\sinh 2k\sigma}{|\cos k\gamma|^2}+\frac{\mathrm{e}^{-3\mathrm{i}k\overline{\gamma}}}{\cos k\overline{\gamma}}-\frac{\mathrm{e}^{-3\mathrm{i}k\gamma}}{\cos k\gamma}\right),$$

the ε-term being needed when $X=0$ because then the imaginary parts of the other terms vanish identically. Thus it is seen that the convergence is an exponential order faster on the imaginary axis itself than on the curve chosen which only approaches the axis asymptotically.

Moreover, as R increases, the decay will quickly become dominated instead by that of the term $g(\tau|\underline{R})$ and this speeds up computations considerably on the curve C. There, when $R>0{\cdot}0001$, the choice used in the routine for the upper truncation value is $Y_{\max}=\min\left(\sigma+\log\frac{30}{R},20\sigma/\cosh^{-1}\left(\frac{k-1}{k-2}\right)\right)$, whilst for integration on the imaginary axis ($k<2{\cdot}3$), this can be relaxed to $Y_{\max}=\sigma+\min(\log\frac{30}{R},20/k)$.

8.2. Output

Examples are presented below where the full solution has been computed through numerical integration on the appropriate contour. One objective of the present study is to provide access to application by the non-specialist mathematician. A single f77 routine (prog3.f for potentials) is thus made available to the reader on the site[1]

$www/lgu.ac.uk/cismres/xtra/prog3.f.$

[1]site address may change with time; search author at: $www/londonmet.ac.uk$

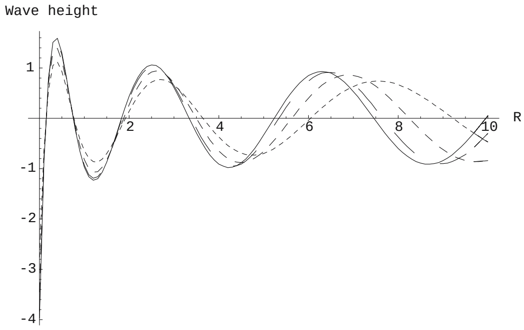

Figure 8. Wave heights for varying incidence; beach angle $\alpha = \pi/31{\cdot}5$; angle of incidence, full line: 1°, broken line 21°, dashed line 41°, dotted line 61°.

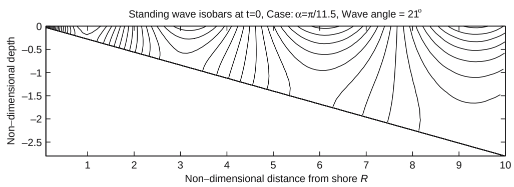

Figure 9. Contours calculated by MATLAB from a (R, θ) -grid: 100×10.

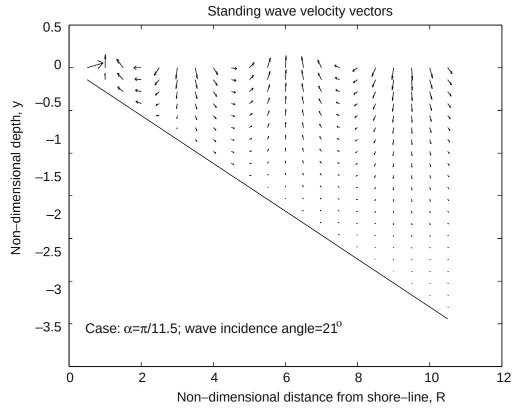

Figure 10. Velocity vectors computed from Equation (31).

(A similar file /prog4.f gives the velocity vectors) The user inputs the beach angle and the desired wave-incidence angle and the wave-form output is written to file (see coding for details). The polar angle may be varied from its null default value to allow also the potential to be computed in the interior of the flow. The choice of contour is automated in the routine.

Here, three different beach angles are examined, namely $\pi/3, \pi/4{\cdot}2$, and $\pi/31{\cdot}5$ and four different design incidence angles, $1^\circ, 21^\circ, 41^\circ$ and 61°, are taken on each of these. The incidence angle 1° allows the comparison with the normal-incidence case. In particular, it allows the verification of the shoreline amplification factor, which is $\surd k$ in the 2-d case. The first example $\alpha = \pi/3$ is chosen as an incline which could be of considerable interest from the point of view of coastal-defence structures but for which, nevertheless, there are no published computations. The results (which have been verified against the unpublished work [7]) are shown in Figure 6.

The second example is chosen as it allows direct comparison with similar work undertaken by the author in [6] for beaches of very simple slope. The same diagram there depicts the situation for the beach $\alpha = \pi/4$ (see Figure 2a therein) and it is noteworthy that, whereas for this result the residue computation at $R = 0$ for example is $\surd 2$, for the beach $\alpha = \pi/4{\cdot}2$ it is over twice as much with the remainder integral establishing the correction to the value $\surd 2{\cdot}1$. This discrepancy becomes more significant as beach angles diminish (as discussed earlier) and could eventually lead to computational difficulties because of the inordinately large cancellation required. Moreover, the accurate cancellation of this discrepancy by the contour integration must be seen as validation of the integration method. The wave profiles are shown in Figure 7. The third example (Figure 8) is chosen because it is a relatively shallow beach. For the first case, the remainder integral is taken along the imaginary axis, whilst, for the other two, the alternative contour is used.

Finally, to demonstrate the f77 routine, also presented for a random case chosen with $\alpha = \pi/11{\cdot}5$ and angle of wave attack 21° are (i) isobars shown in Figure 9 and (ii) velocity vectors shown in Figure 10.

9. Concluding remarks

This work has highlighted, for the 3-d plane-beach-scattering problem, a scarcity of computational techniques for evaluating potentials and/or velocities in all but the most simple of cases. A new way of developing the basic solution for the bounded standing wave has been proved rigorously and explored numerically to demonstrate that a robust method exists when the beach and wave incidence angles are acute but otherwise totally arbitrary. When beach angles are steep (here chosen arbitrarily by the condition $k < 2{\cdot}3$), integration can be taken on the imaginary axis, but for greater non-integer values of k (notably $k > 8$) the contribution from the residues becomes large and requires cancellation with the contribution from the integral along the imaginary axis. This is a classical case of numerical interference and is overcome by using instead the alternative contour which requires only the residues from the two principal poles.

The validity of the procedure of computations relies on three essential tests: (i) that the computed potentials are in agreement with those already known for simple beaches [6], (ii) that the computed potentials are in agreement with those calculated by Bruce [7] for the non-simple beach $\alpha = \pi/3$, and (iii) that the shore-line amplification factor approaches the 2-d value $\sqrt{k}$ for all beach slopes as the incidence angle becomes small. The value of this last test was emphasised by demonstrating that, for non-simple beaches, the residues do not contribute most of this value and so, if the contour-integration technique was somehow unreliable, it would be impossible to get the correct values at the shore line. Meanwhile, the numerical integration packages from the QUADPACK suite are well known to behave robustly, whilst the application of the W-transform to integrals of the type considered here is known to induce relative errors of order 10^{-12} when 10 or more iterations are used (see [27] or [28] for more complete details and illustrative examples).

Whilst the singular (unbounded) standing wave has not been considered here, it is clear that further work is required to establish also a robust computational technique for this wave. This, of course, is required in a model that provides for progressing wave behaviour at infinity (see *e.g.* [29] for the pioneering theory or [6] for computation of this wave on 'simple' beaches). It was hinted earlier in this work and discussed in [6] for simple beaches, that the description of this wave could be achieved by replacing $A(s)$ by $A^*(s)\coth \pi s$, where A^* is of odd parity in Equation (5). The complication introduced by this replacement is first seen in the equivalent to Equation (21) in which the pair of exponential functions in the integrand would be replaced by a similar pair together with an infinite integral that would also need computation before evaluating the outer integral. The reader is referred to [6] for more details (in particular Equation (5.4) and Sections 7 and 8 therein). It remains to be seen, in future work, whether this extra integration prohibits full computation in a reasonable time and therefore whether other approximation techniques need to be explored (such as the near-field expansion discussed in [6, Section 8] supported by a far-field asymptotic expansion). This is considered beyond the scope of the present work.

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Appendix A. The solution of Lauwerier

The solution given in [5] needs to be more fully discussed since it is related to the way the solution is presented here.

The expression, given as Equation (6.7) in Lauwerier's work (L), is written here (for the bounded wave only) in terms of parameters used in the present paper (note that L set $\kappa = 1$ effectively to non-dimensionalise). Thus, only for values on the surface,

$$\phi = 2\mathrm{i} \fint_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\kappa R \sinh u} \, \frac{\cosh u \cosh k\left(u - \frac{1}{2}\pi \mathrm{i}\right)}{\cosh u - \cosh\sigma} \left\{ \frac{2(\cosh u - \cosh\sigma)}{(\cosh 2ku - \cosh 2k\sigma)(\cosh u + \cosh\sigma)} \right\}^{\frac{1}{2}} \times \varphi(u) \; \mathrm{d}u, \tag{A1}$$

where

$$\varphi(u) = \exp\left(\mathrm{i} \int_0^{\infty} \frac{\sin ut}{t} \frac{\cos\sigma t \sinh(\pi/2 - \alpha)t}{\sinh \alpha t \cosh \pi t/2} \; \mathrm{d}t \right).$$

Similarities are noted between φ and Maliuzhinets's function $M_\beta(s)$ and, with the common denominator enjoyed by the two integrands, one could express φ in terms of M_β functions. However, the presentation here will be restricted to demonstrating that the exact solution for the unit-gradient beach can be recovered from L. Indeed, whilst L could be forgiven for not pursuing a computation bearing in mind the limited computing power available half a century ago, it is more difficult to accept that a simple analytic solution was not verified given that Roseau [4] had long ago established one such.

Setting $\alpha = \pi/4$ there follows readily

$$\varphi(u) = \exp\left\{ \mathrm{i} \tan^{-1}\left(\frac{\sinh u}{\cosh\sigma} \right) \right\}$$

and using this, whilst making the transformation $x = \sinh u$ in the expression for ϕ, we achieve a substantial simplification:

$$\phi = 2\mathrm{i} \fint_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\kappa R x} (\mathrm{i}x + \cosh\sigma) \left(\frac{1}{x^2 - \sinh^2\sigma} + \frac{1}{x^2 + \cosh^2\sigma} \right) \, \mathrm{d}x$$

The closed-form expression, given (for all depths) toward the end of Section 5.2, can now be recovered in the usual way, by replacing the principal value (p.v.) integral by the (suitably indented) contour integral and taking account of $2\pi\mathrm{i}$ times the residue at $x = -\mathrm{i}\cosh\sigma$ and $\pi\mathrm{i}$ times the residues at $x = \pm\sinh\sigma$. Meanwhile, the problem of a more general numerical computation from the integral expression requires also replacing the p.v. integral by the field at $R = \infty$ (arising from the residues on the real axis) and an integral along a path in the lower half-plane parallel to the real axis ($\mathfrak{Im}(x) = -\rho$). This could then be the procedure adopted for non-simple beach slopes but there would be complications. In particular, the convergence of the line integral developed from (A1) would not be uniform in any interval including the shore line despite an interpretation of the type $\lim_{X\to\infty} \int_{-X-\mathrm{i}\rho}^{X-\mathrm{i}\rho}$. This would inevitably lead to numerical difficulties in computing values near the shore line. Moreover, branch cuts of the integrand require $\pi/k > \rho > 0$, so choosing say, $\rho = \pi/2k$, we have that the exponential decay of the integrand is only of the order $\mathrm{e}^{-\kappa R \cosh u \sin\alpha}$. This compares poorly with the decay of the integrand used in the model of the present paper which is of the order $\mathrm{e}^{-\kappa R \cosh u \cos\alpha - \nu u}$ when the alternative contour is used and of the order $\mathrm{e}^{-\kappa R \cosh u \cos\alpha - 2ku}$ when the integration is on the imaginary axis ($k < 2{\cdot}3$) Here $\nu = \frac{1}{\sigma} \cosh^{-1} \frac{k-1}{k-2}$. A comparison of the convergence properties of models is shown in Table 1 which x is the integration variable and $\rho(R, x) =$

Table 1. Comparison with Lauwerier's analysis.

Model	Algebraic Decay	Exponential Decay exponent	Oscillatory Wave Number	Convergence
Lauwerier	x^{-1}	0	κR	CPV/Conditional
Modified Lauwerier		$-\kappa R\sin\alpha\cosh x$	$\kappa R\cos\alpha\sinh x$	Absolute if $R>0$
Present (shallow beach)		$-\rho(R,x)-\nu x$	$\kappa R\sin\alpha\sinh x$	Absolute
Present (steep beach)		$-\rho(R,x)-2kx$	0	Absolute

$\kappa R\cos\alpha\cosh x$. It is concluded that Lauwerier's solution is inferior in several respects to the present development.

In practical terms this means that the major difference between Lauwerier's solution and the present solution would seem to lie (i) in the computation of the integrals for small values of R because of exponential decay and also for shallow beaches (ii) in the high $\mathrm{O}(R)$ oscillatory component in Lauwerier, which is only $\mathrm{O}(\alpha R)$ in the present model. Moreover, the exponential decays are then respectively $\mathrm{O}(\alpha R)$ and $\mathrm{O}(R)$ which also favours the present model.

Appendix B. Solution using Barnes's double gamma function

Writing the difference equation (9) in the form

$$\frac{f(\tau+\alpha,\gamma)}{f(\tau-\alpha,\gamma)}=-\frac{\sin\frac{1}{2}(\tau-\gamma)\sin\frac{1}{2}(\tau-(\pi-\gamma))}{\sin\frac{1}{2}(\tau+\gamma)\sin\frac{1}{2}(\tau+(\pi-\gamma))}$$

one can write the solution in the form

$$f(\tau,\gamma)=C(\tau)\mathrm{e}^{\mathrm{i}\pi\tau/2\alpha}\frac{G\left\{\frac{\alpha+2\pi+\gamma-\tau}{2\alpha}\middle|\frac{\pi}{\alpha}\right\}}{G\left\{\frac{\alpha+2\pi-\gamma-\tau}{2\alpha}\middle|\frac{\pi}{\alpha}\right\}}\frac{G\left\{\frac{\alpha+\gamma+\tau}{2\alpha}\middle|\frac{\pi}{\alpha}\right\}}{G\left\{\frac{\alpha-\gamma+\tau}{2\alpha}\middle|\frac{\pi}{\alpha}\right\}}\frac{G\left\{\frac{\alpha+3\pi-\gamma-\tau}{2\alpha}\middle|\frac{\pi}{\alpha}\right\}}{G\left\{\frac{\alpha+\pi+\gamma-\tau}{2\alpha}\middle|\frac{\pi}{\alpha}\right\}}\frac{G\left\{\frac{\alpha+\pi-\gamma+\tau}{2\alpha}\middle|\frac{\pi}{\alpha}\right\}}{G\left\{\frac{\alpha-\pi+\gamma+\tau}{2\alpha}\middle|\frac{\pi}{\alpha}\right\}},\tag{B1}$$

where $G(z,\varpi)$ is Barnes's double gamma function (see [14]) and $C(\tau)$ is periodic of period 2α but is otherwise arbitrary. The double gamma function satisfies two fundamentally important difference relationships namely,

$$G\{z+1|\varpi\}=\Gamma(z/\varpi)G\{z|\varpi\}\tag{B2}$$

and

$$G\{z+\varpi|\varpi\}=(2\pi)^{\frac{1}{2}(\varpi-1)}\varpi^{\frac{1}{2}-z}\Gamma(z)G\{z|\varpi\}\tag{B3}$$

Thus by using (B3) on each of the last pair in the numerator of (B1) and writing, for convenience,

$$\Lambda(\tau)=C(\tau)\mathrm{e}^{\mathrm{i}\pi\tau/2\alpha}(2\pi)^{\pi/\alpha-1}(\pi/\alpha)^{\gamma/\alpha}\Gamma\left(\frac{1}{2}+\frac{\pi-\gamma-\tau}{2\alpha}\right)\Gamma\left(\frac{1}{2}-\frac{\pi+\gamma-\tau}{2\alpha}\right),$$

we obtain alternatively

$$f(\tau,\gamma)=\Lambda(\tau)\frac{G\left\{\frac{\alpha+2\pi+\gamma-\tau}{2\alpha}\middle|\frac{\pi}{\alpha}\right\}}{G\left\{\frac{\alpha+2\pi-\gamma-\tau}{2\alpha}\middle|\frac{\pi}{\alpha}\right\}}\frac{G\left\{\frac{\alpha+\gamma+\tau}{2\alpha}\middle|\frac{\pi}{\alpha}\right\}}{G\left\{\frac{\alpha-\gamma+\tau}{2\alpha}\middle|\frac{\pi}{\alpha}\right\}}\frac{G\left\{\frac{\alpha+\pi-\gamma-\tau}{2\alpha}\middle|\frac{\pi}{\alpha}\right\}}{G\left\{\frac{\alpha+\pi+\gamma-\tau}{2\alpha}\middle|\frac{\pi}{\alpha}\right\}}\frac{G\left\{\frac{\alpha-\pi-\gamma+\tau}{2\alpha}\middle|\frac{\pi}{\alpha}\right\}}{G\left\{\frac{\alpha-\pi+\gamma+\tau}{2\alpha}\middle|\frac{\pi}{\alpha}\right\}}.\tag{B4}$$

The advantage with this representation is that the algebraic sum of the principal arguments is now $(2+\pi/\alpha)$ in both the numerator and denominator of (B4).

One of the many expressions written by Barnes for the double gamma function is

$$G\{z|\varpi\} = \frac{(2\pi\varpi)^{z/2}}{\varpi\Gamma(z)}\varpi^{(z-z^2)/2\varpi}\mathrm{e}^{-zC_1(\varpi)-z^2D(\varpi)/2}\prod_{m=1}^{\infty}\left\{\frac{\Gamma(m\varpi)}{\Gamma(z+m\varpi)}\mathrm{e}^{z\psi(m\varpi)+z^2\psi'(m\varpi)/2}\right\} \quad \text{(B5)}$$

where C_1, D are specified functions of ϖ and ψ is the usual digamma function with a dash (there) denoting a derivative w.r.t its argument. When Equation (B5) is substituted in (B4), all exponential terms linear in z cancel and the quadratic terms greatly simplify. Write, for convenience

$$\Omega(\varpi) = -D(\varpi) + \sum_{m=1}^{\infty}\psi'(m\varpi) - \varpi^{-1}\log\varpi,$$

so that G, Λ in (B4) may effectively be replaced by G^*, Λ^* where

$$G^*\{z|\varpi\} = \frac{1}{\Gamma(z)}\prod_{m=1}^{\infty}\left\{\frac{\Gamma(m\varpi)}{\Gamma(z+m\varpi)}\right\},\ \Lambda^* = \Lambda\mathrm{e}^{4\pi\gamma\Omega(\varpi)},$$

thus considerably reducing the complexity of the solution. After substituting the above and simplifying, we obtain the final expression (17) given in the main text, noting that an arbitrary function of period 2α may always be multiplied on.

Appendix C. Some further technical details

C.1. The value of $|\Omega|$

Setting $\rho = \gamma/\alpha = k + \mathrm{i}\sigma/\alpha$ in Equation (24), we obtain

$$B_k(1)/B_k(\gamma/\alpha) = \exp\left\{-\mathrm{i}\int_0^{\infty}\frac{\sin(\frac{\sigma t}{\alpha})\sinh(k-1+\frac{\mathrm{i}\sigma}{\alpha})t}{t\cosh kt\sinh t}\,\mathrm{d}t\right\}$$

or

$$|B_k(1)/B_k(\gamma/\alpha)| = \exp\int_0^{\infty}\frac{\cosh(k-1)t\sin^2\left(\frac{\sigma t}{\alpha}\right)}{t\cosh kt\sinh t}\,\mathrm{d}t$$
$$= \exp\int_0^{\infty}\frac{\sin^2\left(\frac{\sigma t}{\alpha}\right)}{t}(\coth t - \tanh kt)\,\mathrm{d}t.$$

Take first the result

$$\int_0^{\infty}\frac{\sin^2\left(\frac{\sigma t}{\alpha}\right)}{t}(\coth t - \tanh t)\,\mathrm{d}t = \log\cosh\left(\frac{\pi\sigma/\alpha}{2}\right).$$

Now note that

$$\int_0^{\infty}\frac{\sin^2\left(\frac{\sigma t}{\alpha}\right)}{t}(\tanh t - \tanh kt)\,\mathrm{d}t = -\frac{1}{2}\log\left\{\frac{k\tanh(\sigma)}{\tanh(k\sigma)}\right\}.$$

Adding these two results, we obtain

$$|B_k(1)/B_k(\gamma/\alpha)| = \cosh k\sigma\left(\frac{\tanh k\sigma}{k\tanh\sigma}\right)^{1/2}.$$

C.2. DERIVATION OF CERTAIN RELATIONS SATISFIED BY $f(\tau,\gamma)$

We establish first the relation

$$\frac{f(\tau,\overline{\gamma})}{\cos\frac{\pi}{2\alpha}(\tau-\gamma)}=\frac{f(\tau,\gamma)}{\cos\frac{\pi}{2\alpha}(\tau-\overline{\gamma})}.$$

Using Equation (11) and Malmstén's result

$$\log\Gamma(z)=\int_0^\infty\left\{\frac{e^{-zt}-e^{-t}}{1-e^{-t}}+(z-1)e^{-t}\right\}\frac{dt}{t};\quad \mathfrak{Re}(z)>0,$$

(see [23, p. 249]), we can obtain

$$\log B(s)=\int_0^\infty\frac{e^{-t}}{t}\left(\frac{e^{st}-1-e^{kt}+e^{(k-s+1)t}}{(e^{kt}+1)(1-e^{-t})}-\frac{1}{2}\right)dt.$$

From this, an expression of the type occurring in Equation (16) is readily expressed in the form

$$\log\frac{B_k(s_1)}{B_k(s_2)}=\int_0^\infty\frac{e^{-t}}{t}\left\{\frac{\cosh(2s_1-k-1)t-\cosh(2s_2-k-1)t}{(1-e^{-2t})\cosh kt}\right\}dt,$$

subject to s_1,s_2 being such that the integral converges. This then directly enables us to develop the result

$$\log\frac{f(\tau,\gamma)}{f(\tau,\overline{\gamma})}=-2i\int_0^\infty\frac{\sin\left(\frac{\sigma t}{\alpha}\right)\sinh\left(\frac{\tau-k\alpha}{\alpha}\right)t}{t\sinh t}dt=-2i\tan^{-1}\left\{\tan k\left(\tau-\frac{\pi}{2}\right)\tanh k\sigma\right\}$$

using of [26, Art. 4.114]. Redirecting the i through the $\tan^{-1}\tanh$ operator and expressing $\tanh^{-1}$ in terms of logarithms, provides the required result.

We next wish to show that

$$\frac{f(-\tau,\gamma)}{\cos\frac{\pi}{2\alpha}(\tau-\gamma)}=\frac{f(\tau,\gamma)}{\cos\frac{\pi}{2\alpha}(\tau+\gamma)}.$$

This is most readily observed through the convolution formula (13) which is valid on the strip $-k<\mathfrak{Re}(s)<1+k$ and is easily extended to other strips by use of the analytic continuation formula (10).

Finally the last of the three results,

$$\frac{f(-\tau,\overline{\gamma})}{\cos\frac{\pi}{2\alpha}(\tau-\gamma)}=\frac{f(\tau,\gamma)}{\cos\frac{\pi}{2\alpha}(\tau+\overline{\gamma})}$$

follows directly from the first two.

C.3. COMPUTING $f(\tau,\gamma)$ ON AN ALTERNATIVE CONTOUR

The difficulties described in the text with carrying out computations for shallow beaches or oblique incidence have been largely attributed to round-off-error problems. In this section we revisit the calculation of $f(\tau,\gamma)$ in the strip $0<\mathfrak{Re}(\tau)<\pi/2$ and in particular on the chosen contour $\tau=X+iY$ where $X=(\pi/2-\alpha)\operatorname{sech}\left(\frac{Y}{\sigma}\cosh^{-1}\frac{k-1}{k-2}\right)$. This contour is contained in the strip $0<\mathfrak{Re}(\tau)<\pi/2-\alpha$ is asymptotic to the imaginary axis and bisects the line joining the residues at $\tau=\pi/2-\alpha$, $\tau=\pi/2-3\alpha$.

The previous development of f-values was for use on the imaginary τ-axis and was relatively straightforward in terms of the B_k function and its subsequent integral expressions described herein. These were, however, limited in validity to strips of width 2α and so accessibility to the entire strip $0 < \mathfrak{Re}(\tau) < \pi/2$ would require repeated analytic continuation using formula (10) with the consequence of varying amounts of cotangent products occurring in different sub-strips. Such a development would be more akin to that achievable through Peters's [2] analysis, the numerical difficulties of which have already been discussed.

A fresh approach is therefore required and we take, as a starting point, instead the infinite-product expansion (17) developed using Barnes's double gamma function. We have

$$f(\tau,\gamma) = \Lambda \prod_{m=0}^{\infty} \frac{\Gamma\left\{\frac{\alpha-\gamma-\tau+(2m+2)\pi}{2\alpha}\right\}}{\Gamma\left\{\frac{\alpha-\gamma-\tau+(2m+3)\pi}{2\alpha}\right\}} \frac{\Gamma\left\{\frac{\alpha+\gamma-\tau+(2m+3)\pi}{2\alpha}\right\}}{\Gamma\left\{\frac{\alpha+\gamma-\tau+(2m+2)\pi}{2\alpha}\right\}} \frac{\Gamma\left\{\frac{\alpha+\gamma+\tau+(2m+1)\pi}{2\alpha}\right\}}{\Gamma\left\{\frac{\alpha-\gamma+\tau+(2m+1)\pi}{2\alpha}\right\}} \times \frac{\Gamma\left\{\frac{\alpha-\gamma+\tau+(2m+2)\pi}{2\alpha}\right\}}{\Gamma\left\{\frac{\alpha+\gamma+\tau+(2m+2)\pi}{2\alpha}\right\}},$$

where, upon using (13), we have

$$\Lambda = \frac{\Gamma\left\{\frac{\alpha-\gamma+\tau}{2\alpha}\right\}}{\Gamma\left\{\frac{\alpha+\gamma+\tau}{2\alpha}\right\}} \frac{\Gamma\left\{\frac{\alpha+\gamma-\tau+\pi}{2\alpha}\right\}}{\Gamma\left\{\frac{\alpha-\gamma-\tau+\pi}{2\alpha}\right\}} = \frac{\pi\Gamma\left\{\frac{\alpha+\gamma-\tau+\pi}{2\alpha}\right\}}{\Gamma\left\{\frac{\alpha+\gamma+\tau}{2\alpha}\right\}\Gamma\left\{\frac{\alpha+\gamma-\tau}{2\alpha}\right\}\Gamma\left\{\frac{\alpha-\gamma-\tau+\pi}{2\alpha}\right\}\cos k(\tau-\gamma)},$$

thus ensuring that all arguments occurring in the gamma functions have non-negative real part on $0 \le \mathfrak{Re}(\tau) \le \pi/2$. This enables the further use of Malmstén's result on the expression for $\log f(\tau,\gamma)$. It is noted that the first quotient of gamma functions represented in Λ contains the singularities in the strip and that the second quotient from the infinite product is extracted simply to retain convergence of the Malmstén integral resulting from summation of the logarithm of the remaining product. Doing this on the above, we produce the result

$$\log\Lambda = \log\frac{\pi}{\cos k(\tau-\gamma)} + \int_0^\infty \frac{1}{t(1-e^{-t})}\left\{e^{-z_1 t} - e^{-z_2 t} - e^{-z_3 t} - e^{-z_4 t} + 3e^{-t} - e^{-2t}\right\} dt,$$

where

$$z_1 = \tfrac{1}{2} + (\gamma-\tau+\pi)/2\alpha, \qquad z_2 = \tfrac{1}{2} + (\gamma+\tau)/2\alpha$$
$$z_3 = \tfrac{1}{2} + (\gamma-\tau)/2\alpha, \qquad z_4 = \tfrac{1}{2} + (-\gamma-\tau+\pi)/2\alpha.$$

A similar treatment applied to the remaining infinite product yields

$$\log\prod_{m=0}^{\infty}(\cdot) = \int_0^\infty \frac{dt}{t(1-e^{-t})(1-e^{-2kt})} \times (\Sigma^+ - \Sigma^-)$$

following summation. Here

$$\Sigma^{\pm} = e^{-t(2\pi-\tau\mp\gamma)} + e^{-t(3\pi-\tau\pm\gamma)} + e^{-t(2\pi+\tau\mp\gamma)} + e^{-t(\pi+\tau\pm\gamma)}.$$

Recombining the two expressions to form $f(\tau,\gamma)$, we achieve a considerable simplification and obtain

$$f(\tau,\gamma) = \frac{1}{\cos k(\tau-\gamma)} \exp\int_0^\infty \frac{dt}{t\sinh\alpha t}\left\{1 - \frac{\cosh t\tau}{\cosh\pi t/2}\cos\sigma t\right\}, \tag{C1}$$

a result which is valid in the extended strip $-k-1 < \mathfrak{Re}\,\tau < k+1$, thus obviating the need for any recurrent use of formula (10). Note also, as a check, that setting $\tau = \pi/2$, after some elementary manipulation we recover the result $f(\pi/2,\gamma) = 1$.

A somewhat more numerically stable formula, albeit one that is only valid in $-\alpha < \Re\mathfrak{e}\,\tau < \alpha$, is similarly

$$f(\tau,\gamma) = \frac{\cos k\tau}{\cos k(\tau-\gamma)} \exp \int_0^\infty \frac{\cosh \tau t}{t \sinh \alpha t} \left\{ 1 - \frac{\cos \sigma t}{\cosh \pi t/2} \right\} \, \mathrm{d}t \tag{C2}$$

with the advantage of the integral decaying as $\tau \to \mathrm{i}\infty$ at least like $|\tau|^{-2}$ by the Riemann-Lebesgue lemma. This formula may be used on the line $\tau = \mathrm{i}y$ if $y > \sigma$. If, on the other hand, $y < \sigma$, we use the alternative formula

$$f(\tau,\gamma) = \frac{\cosh k\sigma}{\cos k(\tau-\gamma)} \exp \int_0^\infty \frac{\cos \sigma t}{t \sinh \alpha t} \left\{ 1 - \frac{\cosh \tau t}{\cosh \pi t/2} \right\} \, \mathrm{d}t. \tag{C3}$$

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